Ejercicio 3 – Extraordinaria 2 – Curso 20/21

Sea $f_\alpha : P_2(\mathbb{R}) \to M_2(\mathbb{R})$ la <u>aplicación lineal</u> definida por

$$f_\alpha(p(x)) = \begin{pmatrix} \alpha p'(0) & p'(0) \\ p(1) - p(0) & -\alpha p(0) \end{pmatrix},$$

donde $p'(x)$ es la derivada de $p(x)$.

a. Calcular la matriz asociada a f_α con respecto a las bases canónicas.
b. Clasificar a partir de la matriz obtenida en a), f_α según el valor del parámetro α.
c. Para $\alpha = 1$, calcular base, dimensión, ecuaciones paramétricas e implícitas de $\mathrm{Im}(f_1)$.

a) $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$

$p'(x) = b + 2cx$

$$f_\alpha(a + bx + cx^2) = \begin{pmatrix} \alpha b & b \\ b + c & -\alpha a \end{pmatrix}$$

$B_c = \{1, x, x^2\}$

$$f_\alpha(1) = f_\alpha(1 + 0x + 0x^2) = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix} \equiv (0, 0, 0, -\alpha)$$

$$f_\alpha(x) = f_\alpha(0 + x + 0x^2) = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \equiv (\alpha, 1, 1, 0)$$

$$f_\alpha(x^2) = f_\alpha(0 + 0x + x^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv (0, 0, 1, 0)$$

$$M_{B_cB_c}(f_\alpha) = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ -\alpha & 0 & 0 \end{pmatrix}$$

b) $$\operatorname{rg}\begin{pmatrix} 0 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ -\alpha & 0 & 0 \end{pmatrix} = \begin{cases} 3 \ \dots & \alpha \neq 0 \\ 2 & \alpha = 0 \end{cases}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -\alpha & 0 & 0 \end{vmatrix} = -\alpha = 0 \iff \alpha = 0$$

Si $\operatorname{rg}(M_{B_cB_c}(f_\alpha)) = \dim(\mathcal{P}_2(\mathbb{R})) \Rightarrow f_\alpha$ es inyectiva

Si $\operatorname{rg}(M_{B_cB_c}(f_\alpha)) = \dim(M_2(\mathbb{R})) \Rightarrow f_\alpha$ es sobreyect.

Si $\operatorname{rg}(M_{B_cB'_c}(f_\alpha)) = \dim(\mathcal{P}_2(\mathbb{R})) \Rightarrow f_\alpha$ es biyect.

$$\dim(\mathcal{P}_2(\mathbb{R})) \neq \dim(M_2(\mathbb{R}))$$

$$\begin{cases} \text{Si } \alpha = 0 \Rightarrow \operatorname{rg}(M_{B_cB'_c}(f_\alpha)) = 2 \neq \dim(\mathcal{P}_2(\mathbb{R})) \neq \dim(M_2(\mathbb{R})) \\ \alpha \neq 0 \Rightarrow \operatorname{rg}(M_{B_cB'_c}(f_\alpha)) = 3 = \dim(\mathcal{P}_2(\mathbb{R})) \Rightarrow f_\alpha \text{ es inyectiva o monomorfismo.} \end{cases}$$

c) $\alpha = 1 \neq 0$

$$M_{B_cB_c}(f_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \rightsquigarrow \operatorname{rg}(M_{B_cB'_c}(f_1)) = 3$$

$$B_{\operatorname{Im}(f)} = \{(0,0,0,-1), (1,1,1,0), (0,0,1,0)\} =$$

$$B_{Im(f)} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$$\dim(Im(f)) = 3$$

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \alpha \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} x &= \beta \\ y &= \beta \\ z &= \beta + \gamma \\ t &= -\alpha \end{aligned}\right\}$$ Ec. paramétricas de $Im(f)$.

$$\text{nº implícitas} = \dim(M_2(\mathbb{R})) - \dim(Im(f)) = 4 - 3 = 1$$

$$x = y \Rightarrow \boxed{x - y = 0}$$

Ec. implícita de $Im(f)$.